

On Fourier multipliers in function spaces with partial Hölder condition and their application to the linearized Cahn-Hilliard equation with dynamic boundary conditions.

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We give relatively simple sufficient conditions on a Fourier multiplier, so that it maps functions with the Hölder property with respect to a part of the variables to functions with the Hölder property with respect to all variables. With the using of these sufficient conditions we prove the solvability in Hölder classes of the initial-boundary value problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. For the solutions of these problems Schauder estimates are obtained.

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1 Introduction.

The starting point for this paper is the paper by O.A.Ladyzhenskaya [1]. (see also [2]). The original idea of the reasonings from [1] and [2], as it was pointed in these papers, is taken from [3], Theorem 7.9.6 and this idea is based on the Littlewood-Paley decomposition. However, papers [1] and [2] deal with more general than in [3] case of anisotropic Hölder spaces. Moreover, paper [1] gives some simple sufficient conditions on a Fourier multiplier to provide bounded mapping with this multiplier in anisotropic Hölder spaces. These conditions can be easily verified in particular problems for partial differential equations as it was demonstrated in [1].

It is important that the sufficient conditions from [1] can be comparatively easily verified in the case when a multiplier is an anisotropic-homogeneous function of degree zero (or it is close to such a function in some sense). And it is also fundamentally important in [1] that the anisotropy of a Hölder space where a multiplier acts must coincide with the anisotropy of the multiplier (see Theorem 1 below).

Though the results of [1] are applicable to a broad class of problems for partial differential equations (as it was pointed out in [1]), they are still not applicable to many problems where the anisotropy of a Hölder space does not coincide with the anisotropy of a multiplier or where one should consider a Hölder space of functions with the Hölder conditions with respect only to a part of independent variables.

Among such problems we first mention "nonclassical" statements connected to so called "Newton polygon" see, for example, [4]-[7]. A particular subclass of such problems is the class of problems for parabolic and elliptic equations with highest derivatives in boundary conditions including the Wentzel conditions. This subclass includes also problems with dynamic boundary conditions- [4], [5], [11]-[17]. As it is well known, such problems for parabolic and elliptic equations are not included in the standard general theory of parabolic boundary value problems - see, for example, [8]-[10].

In this paper we apply results about Fourier multipliers in spaces of functions with partial Hölder condition to initial-boundary value problems for linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. To our knowledge earlier such problems were under investigation only in spaces of functions with integrable derivatives - see, for example, [4], [5], [11]-[17]. Not that at the considering of such problems we have the situation when the anisotropy of a multiplier does not coincide with the anisotropy of the corresponding Hölder space. In this situation we consider smoothness of functions with respect to each of their variables separately.

Note also that problems with dynamic boundary conditions arise as a linearization of many well-known free boundary problems such as the Stefan problem, the two-phase filtration problem for two compressible fluids (the parabolic version of the Muskat-Verigin problem), the Hele-Shaw problem, the classical evolutionary Muskat-Verigin problem for elliptic equations.

Besides, the studying of smoothness of solutions of some problems with respect to only a part of independent variables (including obtaining corresponding Schauder's estimates) has it's own history and it is an important direction of investigations. In particular, we deal with such situation when considering semigroups of operators with parameter $t > 0$ and with a generator defined on some Hölder space see, for example, [18]- [34].

In all such cases we can use a theorem about multipliers in spaces of functions with smoothness with respect to only a part of independent variables because it permits to consider the smoothness with respect to each variable separately.

Let us introduce now some notation and formulate an assertion, which is a simple consequence of Theorem 2.1 and lemmas 2.1, 2.2 from [1] because we need it for references.

Let for a natural number N

$$\gamma \in (0, 1), \alpha = (\alpha_1, \alpha_2, \dots, \alpha_N), \quad \alpha_1 = 1, \alpha_k \in (0, 1], k = \overline{2, N}. \quad (1.1)$$

Denote by $C^{\gamma\alpha}(R^N)$ the space of continuous in R^N functions $u(x)$ with the finite norm

$$\|u\|_{C^{\gamma\alpha}(R^N)} \equiv |u|_{R^N}^{(\gamma\alpha)} = |u|_{R^N}^{(0)} + \sum_{i=1}^N \langle u \rangle_{x_i, R^N}^{(\gamma\alpha_i)}, \quad (1.2)$$

where

$$|u|_{R^N}^{(0)} = \sup_{x \in R^N} |u(x)|, \quad \langle u \rangle_{x_i, R^N}^{(\gamma \alpha_i)} = \sup_{x \in R^N, h > 0} \frac{|u(x_1, \dots, x_i + h, \dots, x_N) - u(x)|}{h^{\gamma \alpha_i}}. \quad (1.3)$$

Along with the spaces $C^{\gamma \alpha}(R^N)$ with the exponents $\gamma \alpha_i < 1$ we consider also spaces $C^{\bar{l}}(R^N)$, where $\bar{l} = (l_1, l_2, \dots, l_N)$, l_i are arbitrary positive non-integers. The norm in such spaces is defined by

$$\|u\|_{C^{\bar{l}}(R^N)} \equiv |u|_{R^N}^{(\bar{l})} = |u|_{R^N}^{(0)} + \sum_{i=1}^N \langle u \rangle_{x_i, R^N}^{(l_i)}, \quad (1.4)$$

$$\langle u \rangle_{x_i, R^N}^{(l_i)} = \sup_{x \in R^N, h > 0} \frac{|D_{x_i}^{[l_i]} u(x_1, \dots, x_i + h, \dots, x_N) - D_{x_i}^{[l_i]} u(x)|}{h^{l_i - [l_i]}}, \quad (1.5)$$

where $[l_i]$ is the integer part of the number l_i , $D_{x_i}^{[l_i]} u$ is the derivative of order $[l_i]$ with respect to the variable x_i from a function u . Seminorm (1.3.02) can be equivalently defined by ([35],[36], [37])

$$\langle u \rangle_{x_i, R^N}^{(l_i)} \simeq \sup_{x \in R^N, h > 0} \frac{|\Delta_{h, x_i}^k u(x)|}{h^{l_i}}, \quad k > l_i, \quad (1.6)$$

where $\Delta_{h, x_i} u(x) = u(x_1, \dots, x_i + h, \dots, x_N) - u(x)$ is the difference from a function $u(x)$ with respect to the variable x_i and with step h , $\Delta_{h, x_i}^k u(x) = \Delta_{h, x_i} (\Delta_{h, x_i}^{k-1} u(x)) = (\Delta_{h, x_i})^k u(x)$ is the difference of power k . Note that functions from the space $C^{\bar{l}}(R^N)$ have also mixed derivatives up to definite orders and all derivatives are Hölder continuous with respect to all variables with some exponents in accordance with ratios between the exponents l_i .

Define further the space $\mathcal{H}^{\gamma \alpha}(R^N) \subset C^{\gamma \alpha}(R^N) \cap L_2(R^N)$ as the closure of functions $u(x)$ from $C^{\gamma \alpha}(R^N)$ with finite supports in the norm

$$\|u\|_{\mathcal{H}^{\gamma \alpha}(R^N)} \equiv \|u\|_{L_2(R^N)} + \sum_{i=1}^N \langle u \rangle_{x_i, R^N}^{(\gamma \alpha_i)}. \quad (1.7)$$

Analogously define the space $\mathcal{H}^{\bar{l}}(R^N)$ with arbitrary positive non-integer l_i with the norm

$$\|u\|_{\mathcal{H}^{\bar{l}}(R^N)} \equiv \|u\|_{L_2(R^N)} + \sum_{i=1}^N \langle u \rangle_{x_i, R^N}^{(l_i)}. \quad (1.8)$$

It was shown in [1] that $|u|_{R^N}^{(0)} \leq \|u\|_{\gamma \alpha(R^N)}$ and so

$$|u|_{R^N}^{(\gamma\alpha)} \leq \|u\|_{\gamma\alpha(R^N)}, \quad (1.9)$$

where here and below we denote by C , ν all absolute constants or constant depending on fixed data only.

Let a function $\tilde{m}(\xi)$, $\xi \in R^N$ be defined in R^N and let it be measurable and bounded. Define the operator $M : \mathcal{H}^{\gamma\alpha}(R^N) \rightarrow L_2(R^N)$ according to the formula

$$Mu \equiv F^{-1}(\tilde{m}(\xi)\tilde{u}(\xi)), \quad (1.10)$$

where for a function $u(x) \in L_1(R^N)$

$$\tilde{u}(\xi) \equiv F(u) = \int_{R^N} e^{-ix\xi} u(x) dx \quad (1.11)$$

is Fourier transform of $u(x)$ and we extend Fourier transform on the space $L_2(R^N)$. Denote by $F^{-1}\tilde{u}(\xi)$ the inverse Fourier transform of the function $\tilde{u}(\xi)$.

Since $u(x) \in L_2(R^N)$ and the function $\tilde{m}(\xi)$ is bounded, the operator M is well defined. We call the function $\tilde{m}(\xi)$ a Fourier multiplier.

Let the set of the variables $(\xi_1, \dots, \xi_N) = \xi$ is represented as a union of r subsets of length N_i , $i = \overline{1, r}$ so that

$$N_1 + \dots + N_r = N, \quad \xi = (y_1, \dots, y_r), \quad y_1 = (\xi_1, \dots, \xi_{N_1}), \dots, y_r = (\xi_{N_1 + \dots + N_{r-1} + 1}, \dots, \xi_N).$$

Let further ω_i , $i = \overline{1, r}$, mean multi-indices of length N_i

$$\omega_1 = (\omega_{1,1}, \dots, \omega_{1,N_1}), \dots, \omega_r = (\omega_{r,1}, \dots, \omega_{r,N_r}), \quad \omega_{i,k} \in \mathbf{N} \cup \{0\},$$

and thy symbol $D_{y_i}^{\omega_i} \tilde{u}(\xi)$ means a derivative of a function $\tilde{u}(\xi)$ of order $|\omega_i| = \omega_{i,1} + \dots + \omega_{i,N_i}$ with respect to the group of variables $y_i = (\xi_{k_1}, \dots, \xi_{k_{N_i}})$, that is $D_{y_i}^{\omega_i} \tilde{u}(\xi) = D_{\xi_{k_1}}^{\omega_{i,1}} \dots D_{\xi_{k_{N_i}}}^{\omega_{i,N_i}} \tilde{u}(\xi)$. Let also $p \in (1, 2]$ and positive integers s_i , $i = \overline{1, r}$, satisfy the inequalities

$$s_i > \frac{N_i}{p}, \quad i = \overline{1, r}. \quad (1.12)$$

Denote for $\nu > 0$

$$B_\nu = \{\xi \in R^N : \nu \leq |\xi| \leq \nu^{-1}\}.$$

Suppose that for some $\nu > 0$ the function $\tilde{m}(\xi)$ satisfies with some $\mu > 0$ uniformly in $\lambda > 0$ the condition

$$\sum_{|\omega_i| \leq s_i} \left\| D_{y_1}^{\omega_1} D_{y_2}^{\omega_2} \dots D_{y_r}^{\omega_r} \tilde{m}(\lambda^{\frac{1}{\alpha_1}} \xi_1, \dots, \lambda^{\frac{1}{\alpha_N}} \xi_N) \right\|_{L_p(B_\nu)} \leq \mu. \quad (1.13)$$

Theorem 1 (*[1] : T.2.1, L.2.1, L.2.2, T.2.2, T.2.3*)

If the function $\tilde{m}(\xi)$ satisfies condition (1.13) then the operator M , which is defined in (1.10), is a linear bounded operator from the space $\mathcal{H}^{\gamma\alpha}(R^N)$ to itself and

$$\|Mu\|_{\mathcal{H}^{\gamma\alpha}(R^N)} \leq C(N, \gamma, \alpha, p, \nu, s_i) \mu \|u\|_{\mathcal{H}^{\gamma\alpha}(R^N)}. \quad (1.14)$$

Note that very often condition (1.13) can be easily verified in applications to differential equations. It is the case when the function $\tilde{m}(\xi)$ is anisotropic homogeneous of degree zero, that is when $\tilde{m}(\lambda^{\frac{1}{\alpha_1}} \xi_1, \dots, \lambda^{\frac{1}{\alpha_N}} \xi_N) = \tilde{m}(\xi)$. Note also that condition (1.13) contains derivatives of the function $\tilde{m}(\xi)$ with respect to variables y_i only up to the order s_i . The case $r = 1$, $N_1 = N$, $p = 2$ is contained in Lemma 2.1 in [1] and Lemma 2.2 in [1] contains the case $r = N$, $N_i = 1$, $s_i = 1$. The general case completely analogous (see the proofs of lemmas 3- 5 below).

The further content of the paper is as follows. In Section 2 we prove a theorem about Fourier multipliers in spaces of functions with Hölder condition with respect to only a part of independent variables. In this section we also give comparatively simple sufficient conditions for the theorem. As a conclusion of the section we demonstrate applications of the theorem about Fourier multipliers by two very simple but interesting in our opinion examples for the Laplace equation and for the heat equation. In section 3, we apply the results of Section 2 to a sketch of an investigation of model problems for a linearized Cahn-Hilliard equation with dynamic boundary conditions of two types.

More detailed investigation of initial-boundary value problems for Cahn-Hilliard equation with dynamic boundary conditions will be given in a forthcoming paper.

2 Theorems about Fourier multipliers in spaces of functions with Hölder condition with respect to a part of their variables

In this section we prove a theorem about Fourier multipliers, which is a generalization of Theorem 2.1 from [1]. The schema of the proof is a modification of corresponding schemas from [1], [3].

Define an anisotropic "distance" ρ in space R^N between points x and y according to the formula

$$\rho(x - y) = \sum_{k=1}^{N-2} |x_k - y_k| + |x_{N-1} - y_{N-1}|^\alpha + |x_N - y_N|^\beta, \quad \alpha \in (0, 1], \beta > 0. \quad (2.1)$$

Let us stress that the exponent $\beta > 0$ is an arbitrary great positive number.

Choose a function $\omega(\rho) : [0, +\infty) \rightarrow [0, 1]$ from the class C^∞ such that $\omega \equiv 1$ on the interval $[1/2, 2]$ and $\omega \equiv 0$ on the set $[0, 1/4] \cup [4, +\infty)$. Denote

$$\chi : R^N \rightarrow [0, 1], \quad \chi(\xi) \equiv \omega(\rho(\xi)), \quad \xi \in R^N.$$

Let a function $\tilde{m}(\xi) \in C(R^N \setminus \{0\})$ be bounded. For $x \in R^N$ and for an integer $j \in Z$ denote

$$A_j x \equiv (2^j x_1, \dots, 2^j x_{N-2}, 2^{\frac{j}{\alpha}} x_{N-1}, 2^{\frac{j}{\beta}} x_N), \quad a_j = \det A_j = 2^{j(N-2) + \frac{j}{\alpha} + \frac{j}{\beta}}. \quad (2.2)$$

Denote further $\tilde{m}_j(\xi) = \tilde{m}(\xi) \chi(A_j^{-1} \xi)$, denote by $m_j(x)$ the inverse Fourier transform of the function $\tilde{m}_j(\xi)$, and denote

$$n_j(x) = a_j^{-1} m_j(A_j^{-1} x). \quad (2.3)$$

For convenience we also denote for $x \in R^N$ the variables $x' = (x_1, \dots, x_{N-2}, x_{N-1})$, $x'' = (x_1, \dots, x_{N-2})$.

Let with some $\mu > 0$ the following conditions are satisfied

$$\tilde{m}(\xi)|_{\xi'=0} = \tilde{m}(0, \dots, 0, \xi_N) \equiv 0, \quad \xi_N \in R^1, \quad (2.4)$$

$$\int_{R^N} (1 + |x''|^\gamma + |x_{N-1}|^{\alpha\gamma}) |n_j(x)| dx \leq \mu, \quad j \in Z. \quad (2.5)$$

Let finally we have a function $u(x) \in C_{x'', x_{N-1}}^{\gamma, \alpha\gamma}(R^N)$ with finite support, that is particularly

$$\langle u \rangle_{x'', R^N}^{(\gamma)} + \langle u \rangle_{x_{N-1}, R^N}^{(\alpha\gamma)} < \infty.$$

Denote

$$v(x) = m(x) * u(x) \equiv F^{-1}(\tilde{m}(\xi) \tilde{u}(\xi)).$$

Theorem 2 *Under conditions (2.4), (2.5) the function $v(x)$ satisfies Hölder conditions with respect to all variables and the following estimate is valid*

$$\langle v \rangle_{x'', R^N}^{(\gamma)} + \langle v \rangle_{x_{N-1}, R^N}^{(\alpha\gamma)} + \langle v \rangle_{x_N, R^N}^{(\beta\gamma)} \leq C\mu \left(\langle u \rangle_{x'', R^N}^{(\gamma)} + \langle u \rangle_{x_{N-1}, R^N}^{(\alpha\gamma)} \right). \quad (2.6)$$

Proof. We will try to retain the notation of [1] where it is possible.

Let $\psi \in C^\infty([0, \infty))$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $[0, 1]$ and $\psi \equiv 0$ on $[2, \infty)$. Denote $\varphi(\rho) = \psi(\rho) - \psi(2\rho)$ for $\rho \in [0, \infty)$. The function $\varphi(\rho)$ possess the properties: $\varphi(\rho) \equiv 0$ on $[0, 1/2]$ and $\varphi(\rho) \equiv 0$ on $[2, \infty)$. Denote further

$$\varphi_j(\rho) = \varphi\left(\frac{\rho}{2^j}\right), \quad \varphi_j : [0, \infty) \rightarrow [0, 1], \quad j \in \mathbb{Z}.$$

This set of the functions satisfies by the definition

$$\sum_{j=-\infty}^{\infty} \varphi_j(\rho) = 1, \quad \rho \in (0, \infty). \quad (2.7)$$

Define functions $\tilde{\Phi}$ and $\tilde{\Phi}_j : R^N \rightarrow [0, 1]$ according to the formulas (ρ is from (2.1))

$$\tilde{\Phi}(\xi) \equiv \varphi \circ \rho(\xi) = \varphi(\rho(\xi)), \quad (2.8)$$

$$\tilde{\Phi}_j(\xi) \equiv \varphi_j \circ \rho(\xi) = \varphi_j(\rho(\xi)) = \varphi\left(\frac{\rho(\xi)}{2^j}\right) = . \quad (2.9)$$

$$= \varphi\left(\rho\left(\frac{\xi''}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}}\right)\right) = \tilde{\Phi}\left(\frac{\xi''}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}}\right). \quad (2.10)$$

By the definition of the functions $\tilde{\Phi}_j$ and in view of (2.7)

$$\sum_{j=-\infty}^{\infty} \tilde{\Phi}_j(\xi) \equiv 1, \quad \xi \in R^N \setminus \{0\}. \quad (2.11)$$

We use this equality to represent the function $\tilde{u}(\xi) = F(u(x))$ as

$$\tilde{u}(\xi) = \sum_{j=-\infty}^{\infty} \tilde{u}_j(\xi), \quad \tilde{u}_j(\xi) = \tilde{u}(\xi) \tilde{\Phi}_j(\xi), \quad \xi \in R^N \setminus \{0\}. \quad (2.12)$$

Denote also

$$\Phi(x) = F^{-1}(\tilde{\Phi}(\xi)), \quad \Phi_j(x) = F^{-1}(\tilde{\Phi}_j(\xi)). \quad (2.13)$$

In view of (2.10)

$$\Phi_j(x) = F^{-1}(\tilde{\Phi}_j(\xi)) = \int_{R^N} e^{ix\xi} \tilde{\Phi}\left(\frac{\xi''}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}}\right) d\xi.$$

Making in the last integral the change of the variables $\xi'' = 2^j \eta''$, $\xi_{N-1} = 2^{j/\alpha} \eta_{N-1}$, $\xi_N = 2^{j/\beta} \eta_N$, we obtain

$$\Phi_j(x) = 2^{(N-2)j + \frac{j}{\alpha} + \frac{j}{\beta}} \Phi(2^j x'', 2^{j/\alpha} x_{N-1}, 2^{j/\beta} x_N) = a_j \Phi(A_j x). \quad (2.14)$$

In view of the above definition of the function $\chi(\xi)$ and in view of the definition of the functions $\tilde{\Phi}(\xi)$ и $\tilde{\Phi}_j(\xi)$ we have for all $\xi \in R^N$

$$\tilde{\Phi}(\xi) = \tilde{\Phi}(\xi)\chi(\xi), \quad \tilde{\Phi}_j(\xi) = \tilde{\Phi}_j(\xi)\chi\left(\frac{\xi''}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}}\right).$$

Consequently,

$$\tilde{u}_j(\xi) = \tilde{u}(\xi)\tilde{\Phi}_j(\xi) = \tilde{u}_j(\xi)\chi\left(\frac{\xi''}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}}\right) \equiv \tilde{u}_j(\xi)\chi_j(\xi), \quad (2.15)$$

where

$$\chi_j(\xi) \equiv \chi\left(\frac{\xi''}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}}\right). \quad (2.16)$$

Denote in the cense of distributions

$$v(x) = m(x) * u(x), \quad (2.17)$$

that is

$$\begin{aligned} \tilde{v}(\xi) &= \tilde{m}(\xi)\tilde{u}(\xi) = \sum_{j=-\infty}^{\infty} \tilde{m}(\xi)\tilde{u}_j(\xi) = \\ &= \sum_{j=-\infty}^{\infty} \tilde{m}(\xi)\chi_j(\xi)\tilde{u}_j(\xi) \equiv \sum_{j=-\infty}^{\infty} \tilde{m}_j(\xi)\tilde{u}_j(\xi) \equiv \sum_{j=-\infty}^{\infty} \tilde{v}_j(\xi), \end{aligned} \quad (2.18)$$

where

$$\tilde{m}_j(\xi) \equiv \tilde{m}(\xi)\chi_j(\xi), \quad \tilde{v}_j(\xi) = \tilde{m}_j(\xi)\tilde{u}_j(\xi). \quad (2.19)$$

Consider the function $v_j(x) = F^{-1}(\tilde{v}_j(\xi))$. Using it's definition and (2.2), we represent this function in the form

$$\begin{aligned} v_j(x) &= u(x) * \Phi_j(x) * m_j(x) = \int_{R^N} u(x-y)dy \int_{R^N} m_j(y-z)\Phi_j(z)dz = \\ &= \int_{R^N} u(x-y)dy \int_{R^N} m_j(y-z)a_j\Phi(A_jz)dz. \end{aligned} \quad (2.20)$$

Making in the last integral the change of the variables $k = A_jz$, $dk = a_jdz$, we obtain

$$v_j(x) = \int_{R^N} u(x-y)dy \int_{R^N} m_j(y - A_j^{-1}k)\Phi(k)dk.$$

Making now the change of the variables $y = A_j^{-1}z$, we arrive at the expression

$$v_j(x) = \int_{R^N} u(x - A_j^{-1}z) dz \int_{R^N} [a_j^{-1} m_j(A_j^{-1}(z - k))] \Phi(k) dk \equiv \int_{R^N} u(x - A_j^{-1}z) \theta_j(z) dz, \quad (2.21)$$

where

$$\theta_j(z) = n_j(z) * \Phi(z) = \int_{R^N} n_j(z - k) \Phi(k) dk, \quad (2.22)$$

and the function $n_j(z)$ is defined in (2.3).

Calculate now the derivatives v_{x_i} , $i = 1, 2, \dots, N$. For this we use properties of a convolution and analogously (2.20) represent $v_j(x)$ as

$$v_j(x) = \int_{R^N} a_j \Phi(A_j(x - y)) dy \int_{R^N} u(y - z) m_j(z) dz.$$

Let first $i = N - 1$. Then

$$(v_j(x))_{x_{N-1}} = 2^{\frac{j}{\alpha}} \int_{R^N} a_j \Phi^{(i)}(A_j(x - y)) dy \int_{R^N} u(y - z) m_j(z) dz,$$

where

$$\Phi^{(i)}(z) \equiv \frac{\partial \Phi}{\partial z_i}(z), \quad i = 1, 2, \dots, N. \quad (2.23)$$

Using properties of a convolution and making a series of changes of variables with the matrices A_j and A_j^{-1} , we obtain completely analogously to (2.21)

$$(v_j(x))_{x_{N-1}} = 2^{\frac{j}{\alpha}} \int_{R^N} u(x - A_j^{-1}z) (\theta_j)_{z_{N-1}}(z) dz, \quad (2.24)$$

where

$$(\theta_j)_{z_{N-1}}(z) = \int_{R^N} n_j(z - k) \Phi_{k_{N-1}}(k) dk.$$

Completely analogously for $1 \leq i \leq N - 2$

$$(v_j(x))_{x_i} = 2^j \int_{R^N} u(x - A_j^{-1}z) (\theta_j)_{z_i}(z) dz, \quad (2.25)$$

and for $i = N$

$$(v_j(x))_{x_N} = 2^{\frac{j}{\beta}} \int_{R^N} u(x - A_j^{-1}z) (\theta_j)_{z_N}(z) dz,$$

and also in more general case

$$D_{x_N}^k (v_j(x)) = 2^{\frac{kj}{\beta}} \int_{R^N} u(x - A_j^{-1}z) D_{z_N}^k (\theta_j)(z) dz, \quad k = 0, 1, 2, \dots$$

Now note that for almost all z_N

$$\begin{aligned} f_j(z_N) &\equiv \int_{R^{N-1}} \theta_j(z', z_N) dz' \equiv 0, \quad f_j^{(i)}(z_N) \equiv \int_{R^{N-1}} (\theta_j)_{z_i}(z', z_N) dz' \equiv 0, \\ \int_{R^{N-1}} D_{z_N}^k (\theta_j)(z', z_N) dz' &\equiv 0. \end{aligned} \quad (2.26)$$

We show the first of these relations as the second and the third are completely similar. It suffices to show that the Fourier transform of $f_j(z_N)$ with respect to z_N is identically equal to zero

$$\begin{aligned} F_N(f_j) &= \tilde{f}_j(\xi_N) = \int_{R^1} e^{-iz_N \xi} dz_N \int_{R^{N-1}} \theta_j(z', z_N) dz' = \\ &= \int_{R^{N-1}} dz' \int_{R^1} e^{-iz_N \xi} \theta_j(z', z_N) dz_N = \int_{R^{N-1}} F_N(\theta_j)(z', \xi_N) dz'. \end{aligned}$$

Since the integral with respect to z' of the function $F_N(\theta_j)(z', \xi_N)$ is equal to the value at $\xi' = 0$ of it's Fourier transform with respect to the same variables z' , then

$$\begin{aligned} \int_{R^{N-1}} F_N(\theta_j)(z', \xi_N) dz' &= \left[\int_{R^{N-1}} e^{-iz' \xi'} F_N(\theta_j)(z', \xi_N) dz' \right] \Big|_{\xi'=0} = \\ &= \tilde{\theta}_j(\xi', \xi_N) \Big|_{\xi'=0} = \tilde{\theta}_j(0, \xi_N) \equiv 0. \end{aligned}$$

The last identity follows from the fact that

$$\tilde{n}_j(\xi) = \int_{R^N} e^{-ix\xi} a_j^{-1} m_j(A_j^{-1}x) dx = \tilde{m}_j(A_j \xi), \quad (2.27)$$

and consequently

$$\tilde{\theta}_j(\xi) = \tilde{n}_j(\xi)\tilde{\Phi}(\xi) = \tilde{m}_j(A_j\xi)\tilde{\Phi}(\xi) = \tilde{m}(A_j\xi)\chi_j(\xi)\tilde{\Phi}(\xi).$$

Therefore in view of (2.4) we have $\tilde{\theta}_j(0, \xi_N) \equiv 0$. Thus the first relation (2.26) is proved. The second is similar.

Let us obtain now the estimates

$$\int_{R^N} |z''|^\gamma |\theta_j(z)| dz \leq C\mu, \quad \int_{R^N} |z_{N-1}|^{\alpha\gamma} |\theta_j(z)| dz \leq C\mu, \quad j \in Z, \quad (2.28)$$

$$\int_{R^N} |z''|^\gamma |(\theta_j)_{z_k}(z)| dz \leq C\mu, \quad \int_{R^N} |z_{N-1}|^{\alpha\gamma} |(\theta_j)_{z_k}(z)| dz \leq C\mu, \quad j \in Z, k = \overline{1, N}, \quad (2.29)$$

$$\int_{R^N} |z''|^\gamma |D_{z_N}^n(\theta_j)(z)| dz \leq C\mu, \quad \int_{R^N} |z_{N-1}|^{\alpha\gamma} |D_{z_N}^n(\theta_j)(z)| dz \leq C\mu, \quad j \in Z, n = 1, 2, \dots, \quad (2.30)$$

where μ is from condition (2.5). We obtain only the first inequality (2.28) because the rest is quite similar. Indeed, if $y'' \in R^{N-2}$ then we use the inequality $|z''|^\gamma \leq C(|y''|^\gamma + |z'' - y''|^\gamma)$ and in view of the definition of θ_j in (2.22) we obtain

$$\begin{aligned} \int_{R^N} |z''|^\gamma |\theta_j(z)| dz &\leq \int_{R^N} |z''|^\gamma dz \int_{R^N} |n_j(y)| |\Phi(z - y)| dy \leq \\ &\leq C \int_{R^N} dz \int_{R^N} [|y''|^\gamma |n_j(y)|] |\Phi(z - y)| dy + C \int_{R^N} dz \int_{R^N} |n_j(y)| [|z'' - y''|^\gamma |\Phi(z - y)|] dy \leq \\ &\leq \int_{R^N} (1 + |y''|^\gamma) |n_j(y)| dy \leq C\mu, \end{aligned}$$

as it follows from properties of the function $\Phi(x)$.

Let us now estimate the Hölder constant of the function $v(x)$. For this we estimate $|v_j(x)|$ and $|(v_j)_{x_i}(x)|$. From (2.21) and (2.26) it follows that

$$\begin{aligned} v_j(x) &= \int_{R^1} dz_N \int_{R^{N-1}} u(x - A_j^{-1}z) \theta_j(z) dz' = \\ &= \int_{R^1} dz_N \int_{R^{N-1}} \left[u(x - A_j^{-1}z) - u(x'', x_{N-1}, x_N - \frac{z_N}{2^{j/\beta}}) \right] \theta_j(z) dz' = \end{aligned}$$

$$= \int_{R^N} \left[u\left(x'' - \frac{z''}{2^j}, x_{N-1} - \frac{z_{N-1}}{2^{j/\alpha}}, x_N - \frac{z_N}{2^{j/\beta}}\right) - u\left(x'', x_{N-1}, x_N - \frac{z_N}{2^{j/\beta}}\right) \right] \theta_j(z) dz. \quad (2.31)$$

From this, in view of the fact that $u(x)$ satisfies the Hölder condition with respect to the first $N - 1$ variables and in view of estimate (2.28), it follows that

$$|v_j(x)| \leq C 2^{-\gamma j} \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \int_{R^N} (|z''|^\gamma + |z_{N-1}|^{\alpha\gamma}) |\theta_j(z)| dz \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{-\gamma j}. \quad (2.32)$$

And for $i = \overline{1, N-2}$ we similarly obtain

$$(v_j)_{x_i}(x) = 2^j \int_{R^N} \left[u\left(x'' - \frac{z''}{2^j}, x_{N-1} - \frac{z_{N-1}}{2^{j/\alpha}}, x_N - \frac{z_N}{2^{j/\beta}}\right) - u\left(x'', x_{N-1}, x_N - \frac{z_N}{2^{j/\beta}}\right) \right] (\theta_j)_{z_i}(z) dz,$$

and therefore we have similarly to (2.32)

$$|(v_j)_{x_i}(x)| \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{j-\gamma j}, \quad i = \overline{1, N-2}. \quad (2.33)$$

Likewise for $i = N - 1$ and for $i = N$

$$|(v_j)_{x_{N-1}}(x)| \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{\frac{j}{\alpha}-\gamma j}, \quad (2.34)$$

$$|(v_j)_{x_N}(x)| \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{\frac{j}{\beta}-\gamma j}, \quad (2.35)$$

and more generally for $i = N$

$$|D_{x_N}^k(v_j)(x)| \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{\frac{kj}{\beta}-\gamma j}, \quad k = 1, 2, \dots \quad (2.36)$$

Let now $x, y \in R^N$. Consider first the case when $\beta\gamma < 1$. We have

$$|v(x) - v(y)| \leq \sum_{j=-\infty}^{\infty} |v_j(x) - v_j(y)| = \sum_{j \geq n_0} |v_j(x) - v_j(y)| + \sum_{j \leq n_0} |v_j(x) - v_j(y)| \equiv S_1 + S_2, \quad (2.37)$$

where $n_0 = -\log_2 \rho(x - y)$. To estimate S_1 we use inequalities (2.32)

$$\begin{aligned} S_1 &\leq \sum_{j \geq n_0} 2|v_j|^{(0)} \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \sum_{j \geq n_0} 2^{-\gamma j} \leq \\ &\leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{-n_0\gamma} \sum_{k=0}^{\infty} 2^{-\gamma k} \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \rho^\gamma(x - y). \end{aligned} \quad (2.38)$$

To estimate S_2 we use the mean value theorem for the difference $|v_j(x) - v_j(y)|$ and estimates (2.33)- (2.35) for the corresponding derivatives

$$\begin{aligned}
S_2 &\leq C \sum_{j \leq n_0} \left(\sum_{k=1}^N |x_k - y_k| |(v_j)_{x_k}|^{(0)} \right) \leq \\
&\leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \sum_{j \leq n_0} \left(|x'' - y''| 2^{j-j\gamma} + |x_{N-1} - y_{N-1}| 2^{\frac{j}{\alpha}-j\gamma} + |x_N - y_N| 2^{\frac{j}{\beta}-j\gamma} \right) \leq \\
&\leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \left(|x'' - y''| 2^{(1-\gamma)n_0} \sum_{k=0}^{\infty} 2^{-(1-\gamma)k} + \right. \\
&\quad \left. + |x_{N-1} - y_{N-1}| 2^{n_0(\frac{1}{\alpha}-\gamma)} \sum_{k=0}^{\infty} 2^{-(\frac{1}{\alpha}-\gamma)k} + |x_N - y_N| 2^{n_0(\frac{1}{\beta}-\gamma)} \sum_{k=0}^{\infty} 2^{-(\frac{1}{\beta}-\gamma)k} \right) \leq \\
&\leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \left(|x'' - y''| \rho^{-1+\gamma}(x-y) + \right. \\
&\quad \left. + |x_{N-1} - y_{N-1}| \rho^{-\frac{1}{\alpha}+\gamma}(x-y) + |x_N - y_N| \rho^{-\frac{1}{\beta}+\gamma}(x-y) \right) \leq \\
&\leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \rho^\gamma(x-y),
\end{aligned} \tag{2.39}$$

where we have used the fact that $\beta\gamma < 1$ and consequently $(\frac{1}{\beta} - \gamma) > 0$.

From (2.38) and (2.39) it follows that

$$|v(x) - v(y)| \leq C \mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} \rho^\gamma(x-y). \tag{2.40}$$

This proves the theorem in the case $\beta\gamma < 1$.

Let now $\beta\gamma > 1$.

The proof in this case requires only a small change. Firstly, selecting in the previous proof point x and y such that $x_N = y_N$, that is considering the Hölder property of the function $v(x)$ only with respect to the variables x' , we obtain estimate (2.39) with $|x_N - y_N| = 0$. This proves (2.40) for such x and y and hence this gives the desired smoothness of $v(x)$ with respect to the variables x' . Now the smoothness property of this function in the variable x_N should be considered separately. For this purpose, with definition (1.6) in mind, we need to consider k -th difference in variable x_N of the function $v(x)$. Let k be a sufficiently large positive integer such that $k/\beta > \gamma$, $h > 0$. Similarly to the previous

$$|\Delta_{h, x_N}^k v(x)| \leq \sum_{j=-\infty}^{\infty} |\Delta_{h, x_N}^k v_j(x)| = \sum_{j \geq n_0} |\Delta_{h, x_N}^k v_j(x)| + \sum_{j \leq n_0} |\Delta_{h, x_N}^k v_j(x)| \equiv S_1 + S_2,$$

where $n_0 = -\log_2 h^\beta$. The sum S_1 is estimated at exactly the same way as above, which gives

$$S_1 \leq C\mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} h^{\beta\gamma}.$$

The sum S_2 is also evaluated as before taking into account the fact that

$$|\Delta_{h, x_N}^k v_j(x)| \leq Ch^k |D_{x_N}^k v_j(x)|^{(0)} \leq C\mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} 2^{\frac{kj}{\beta} - \gamma j},$$

where we also used estimate (2.36). This gives

$$S_2 \leq C\mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} h^k 2^{n_0(\frac{k}{\beta} - \gamma)} \sum_{k=0}^{\infty} 2^{-(\frac{k}{\beta} - \gamma)k} \leq C\mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} h^{\beta\gamma}.$$

From the estimates for S_1 and S_2 it follows that

$$|\Delta_{h, x_N}^k v(x)| \leq C\mu \langle u \rangle_{x'}^{(\gamma, \alpha\gamma)} h^{\beta\gamma}.$$

Thus by definition (1.6) the theorem is proved. ■

Following the idea of [1] and similar to the conditions of Theorem 1, we give simple sufficient conditions on $\tilde{m}(\xi)$ to have condition (2.5). Note first that after the change of the variables $y = A_j^{-1}x$ we obtain

$$\tilde{n}_j(\xi) = C \int_{R^N} e^{ix\xi} a_j^{-1} m_j(A_j^{-1}x) dx = C \int_{R^N} e^{i(y, A_j\xi)} m_j(y) dy = \tilde{m}_j(A_j\xi) = \tilde{m}(A_j\xi) \chi(\xi). \quad (2.41)$$

Denote for $\lambda > 0$ $A_\lambda\xi = (\lambda\xi'', \lambda^{\frac{1}{\alpha}}\xi_{N-1}, \lambda^{\frac{1}{\beta}}\xi_N)$ and denote $B_0 = \{\xi \in R^N : 1/8 \leq \rho(\xi) \leq 8\}$. All sufficient conditions to have (2.5), which we state below, are linked with the property of the Fourier transform

$$-ix_k f(x) = \widetilde{f_{\xi_k}},$$

as well as with the well-known the Hausdorff-Young inequality

$$\|f(x)\|_{L_{p'}(R^N)} \leq C_{N,p} \left\| \widetilde{f}(\xi) \right\|_{L_p(R^N)}, \quad p \in (1, 2], \quad p' = \frac{p}{p-1}. \quad (2.42)$$

Lemma 3 *Let uniformly in $\lambda > 0$*

$$\tilde{m}(A_\lambda\xi) \in W_p^s(B_0), \quad p \in (1, 2], \quad s > \frac{N}{p} + \gamma.$$

Then conditions (2.5) are satisfied and

$$\mu \leq \sup_{\lambda} C \|\tilde{m}(A_\lambda\xi)\|_{W_p^s(B_0)}. \quad (2.43)$$

Proof. (compare [1]).

In view of (2.41) for $r > N/p$

$$\begin{aligned}
\int_{R^N} (1 + |x''|^\gamma + |x_{N-1}|^{\alpha\gamma}) |n_j(x)| dx &\leq C \int_{R^N} (1 + x^2)^{\frac{\gamma+r}{2}} |n_j(x)| (1 + x^2)^{-\frac{r}{2}} dx \leq \\
&\leq C \left(\int_{R^N} \left[(1 + x^2)^{\frac{\gamma+r}{2}} |n_j(x)| \right]^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{R^N} (1 + x^2)^{-\frac{rp}{2}} dx \right)^{\frac{1}{p}} \leq \\
&\leq C \left[\int_{R^N} \left(\sum_{|\omega|=0}^{\gamma+r} |D_\xi^\omega \tilde{n}_j(\xi)|^p \right) d\xi \right]^{\frac{1}{p}} \leq C \|\tilde{m}(A_j \xi)\|_{W_p^{\gamma+r}(B_0)}.
\end{aligned}$$

The lemma follows. ■

The same idea that was used in the proof of Lemma 3 can be used by groups of the variables. That is, for example, we obtain with $r > (N-1)/p$

$$\begin{aligned}
&\int_{R^N} (1 + |x''|^\gamma + |x_{N-1}|^{\alpha\gamma}) |n_j(x)| dx \leq C \int_{R^N} (1 + (x')^2)^{\frac{\gamma}{2}} |n_j(x)| dx = \\
&= C \int_{R^N} \left[(1 + (x')^2)^{\frac{\gamma+r}{2}} (1 + ix_N) |n_j(x)| \right] \left[(1 + (x')^2)^{-\frac{r}{2}} (1 + ix_N)^{-1} \right] dx \leq \\
&\leq C \left\{ \int_{R^N} \left[(1 + (x')^2)^{\frac{\gamma+r}{2}} (1 + ix_N) |n_j(x)| \right]^{p'} dx \right\}^{\frac{1}{p'}} \left\{ \int_{R^N} (1 + (x')^2)^{-\frac{rp}{2}} (1 + ix_N)^{-p} dx \right\}^{\frac{1}{p}} \leq \\
&\leq C \sum_{\substack{|\omega'| \leq \gamma+r, \\ \omega_N \in \{0,1\}}} \left\| D_{\xi', \xi_N}^{(\omega', \omega_N)} \tilde{m}(A_j \xi) \right\|_{L_p(B_0)},
\end{aligned}$$

where the sum in the last expression is considered in all multi-indices $\omega = (\omega', \omega_N)$ such that $|\omega'| \leq \gamma + r$ with $r > (N-1)/p$ and $\omega_N \in \{0, 1\}$.

Thus the following lemma holds.

Lemma 4 *Let for any $\lambda > 0$ and for some $p \in (1, 2]$ with $s > (N-1)/p$ we have*

$$M_1 \equiv \sup_{\lambda > 0} \sum_{\substack{|\omega'| \leq \gamma+s, \\ \omega_N \in \{0,1\}}} \left\| D_{\xi', \xi_N}^{(\omega', \omega_N)} \tilde{m}(A_\lambda \xi) \right\|_{L_p(B_0)} < \infty.$$

Then condition (2.5) is satisfied and $\mu \leq CM_1$.

Formulate for example yet another assertion, which can be proved exactly the same way as the previous two lemmas taking into account that $(1 + |x''|^\gamma + |x_{N-1}|^{\alpha\gamma}) \leq \prod_{k=1}^{N-1} |1 + ix_k|$ and using the multiplication and division by $\prod_{k=1}^N |1 + ix_k|$.

Lemma 5 *Suppose that for some $p \in (1, 2]$ the following condition is satisfied*

$$M_2 \equiv \sup_{\lambda > 0} \sum_{\omega} \|D_\xi^\omega \tilde{m}(A_\lambda \xi)\|_{L_p(B_0)} < \infty,$$

where the sum is taken over all multi-indices $\omega = (\omega_1, \dots, \omega_{N-1}, \omega_N)$ such that $\omega_1, \dots, \omega_{N-1} \in \{0, 1, 2\}$, $\omega_N \in \{0, 1\}$.

Then condition (2.5) is satisfied and $\mu \leq CM_2$.

The fact that considered in Theorem 2 multiplier $\tilde{m}(\xi)$ uses the smoothness of the density $u(x)$ for all variable x' except for one variable x_N is insignificant and was considered only for simplicity. Directly from the proof of Theorem 2 it follows that completely analogous to this proof the following assertion can be proved.

Let a function $\tilde{m}(\xi) \in C(R^N \setminus \{0\})$ be bounded. Let $x \in R^N$, let $K \in (0, N)$ be an integer, $x = (x^{(1)}, x^{(2)})$, $x^{(1)} = (x_1, \dots, x_K)$, $x^{(2)} = (x_{K+1}, \dots, x_N)$ and similarly $\xi = (\xi^{(1)}, \xi^{(2)})$, $\xi^{(1)} = (\xi_1, \dots, \xi_K)$, $\xi^{(2)} = (\xi_{K+1}, \dots, \xi_N)$. Let $\alpha = (\alpha_1, \dots, \alpha_K)$, $\beta = (\beta_{K+1}, \dots, \beta_N)$, $\alpha_i \in (0, 1]$, $\beta_k > 0$, and $\gamma \in (0, 1)$.

Denote for $x \in R^N$ and for an integer $j \in Z$

$$A_j x \equiv (2^{\frac{j}{\alpha_1}} x_1, \dots, 2^{\frac{j}{\alpha_K}} x_K, 2^{\frac{j}{\beta_{K+1}}} x_{K+1}, 2^{\frac{j}{\beta_N}} x_N), \quad a_j = \det A_j. \quad (2.44)$$

Denote as above $\tilde{m}_j(\xi) = \tilde{m}(\xi) \chi(A_j^{-1} \xi)$, and let $m_j(x)$ be the inverse Fourier transform of the function $\tilde{m}_j(\xi)$,

$$n_j(x) = a_j^{-1} m_j(A_j^{-1} x). \quad (2.45)$$

Let with some $\mu > 0$ the following conditions are satisfied

$$\tilde{m}(\xi)|_{\xi^{(1)}=0} = \tilde{m}(0, \xi^{(2)}) \equiv 0, \quad \xi^{(2)} \in R^{N-K}, \quad (2.46)$$

$$\int_{R^N} (1 + \sum_{k=1}^K |x_k|^{\alpha_k \gamma}) |n_j(x)| dx \leq \mu, \quad j \in Z. \quad (2.47)$$

Suppose finally that a function $u(x) \in C_{x^{(1)}}^{\alpha\gamma}(R^N)$ has a finite support in R^N and satisfies Hölder condition with respect to a part of the variables

$$\langle u \rangle_{x^{(1)}, R^N}^{(\alpha\gamma)} = \sum_{k=1}^K \langle u \rangle_{x_k, R^N}^{(\alpha_k \gamma)} < \infty.$$

Denote as above

$$v(x) \equiv Mu \equiv m(x) * u(x) \equiv F^{-1}(\tilde{m}(\xi)\tilde{u}(\xi)).$$

Theorem 6 *Let conditions (2.46), (2.47) are satisfied. Then the function $v(x)$ satisfies the Hölder condition with respect to all variables and*

$$\langle v \rangle_{x^{(1)}, x^{(2)}, R^N}^{(\alpha\gamma, \beta\gamma)} \leq C\mu \langle u \rangle_{x^{(1)}, R^N}^{(\alpha\gamma)}, \quad (2.48)$$

where

$$\langle v \rangle_{x^{(1)}, x^{(2)}, R^N}^{(\alpha\gamma, \beta\gamma)} = \sum_{k=1}^K \langle v \rangle_{x_k, R^N}^{(\alpha_k\gamma)} + \sum_{k=K+1}^N \langle v \rangle_{x_k, R^N}^{(\beta_k\gamma)}.$$

Completely analogous to the proof of Lemmas 3, 4 a sufficient condition for inequalities (2.47) can be obtained. Similarly with Lemma 3 we have the following assertion.

For $\lambda > 0$ denote $A_\lambda \xi = (\lambda^{\frac{1}{\alpha_1}} \xi_1, \dots, \lambda^{\frac{1}{\alpha_K}} \xi_K, \lambda^{\frac{1}{\beta_{K+1}}} \xi_{K+1}, \dots, \lambda^{\frac{1}{\beta_N}} \xi_N)$ and denote $B_0 = \{\xi \in R^N : 1/8 \leq \rho(\xi) \leq 8\}$, where $\rho(\xi) = \sum_{k=1}^K |\xi_k|^{\alpha_k} + \sum_{k=K+1}^N |\xi_k|^{\beta_k}$.

Lemma 7 *Let uniformly in $\lambda > 0$*

$$\tilde{m}(A_\lambda \xi) \in W_p^s(B_0), \quad p \in (1, 2], \quad s > \frac{N}{p} + \gamma.$$

Then conditions (2.47) are satisfied and

$$\mu \leq \sup_{\lambda} C \|\tilde{m}(A_\lambda \xi)\|_{W_p^s(B_0)}.$$

We also have a more general assertion similar to Theorem 1.

Denote similarly to the previous section the spaces

$$\mathcal{H}_{x^{(1)}}^{\alpha\gamma}(R^N) = C_{x^{(1)}}^{\alpha\gamma}(R^N) \cap L_2(R^N), \quad \mathcal{H}_{x^{(1)}, x^{(2)}}^{\alpha\gamma, \beta\gamma}(R^N) = C_{x^{(1)}, x^{(2)}}^{\alpha\gamma, \beta\gamma}(R^N) \cap L_2(R^N),$$

which are the closures of the set of finite functions in the norms

$$\|u\|_{\mathcal{H}_{x^{(1)}}^{\alpha\gamma}(R^N)} \equiv \|u\|_{L_2(R^N)} + \langle u \rangle_{x^{(1)}, R^N}^{(\alpha\gamma)}, \quad \|u\|_{\mathcal{H}_{x^{(1)}, x^{(2)}}^{\alpha\gamma, \beta\gamma}(R^N)} \equiv \|u\|_{L_2(R^N)} + \langle u \rangle_{x^{(1)}, x^{(2)}, R^N}^{(\alpha\gamma, \beta\gamma)}.$$

Closure of estimate (2.48), proved for a finite function $u(x)$, in the norms of the spaces $\mathcal{H}_{x^{(1)}}^{\alpha\gamma}(R^N)$, $\mathcal{H}_{x^{(1)}, x^{(2)}}^{\alpha\gamma, \beta\gamma}(R^N)$ and the using of the scheme of the proofs of Lemmas 3, 4 leads to the following assertin.

Theorem 8 *Let condition (2.46) be satisfied. Let further in the notation of Theorem 1 instead of condition (1.12) the following condition be satisfied*

$$s_i > \frac{N_i}{p} + \gamma, \quad i = \overline{1, r}, \quad p \in (1, 2]. \quad (2.49)$$

Let also similar to (1.13) the following condition be satisfied

$$\sup_{\lambda > 0} \sum_{|\omega_i| \leq s_i} \|D_{y_1}^{\omega_1} D_{y_2}^{\omega_2} \dots D_{y_r}^{\omega_r} \tilde{m}(A_\lambda \xi)\|_{L_p(B_\nu)} \leq \mu. \quad (2.50)$$

Then the operator M is a bounded linear operator from $\mathcal{H}_{x^{(1)}}^{\alpha\gamma}(R^N)$ to $\mathcal{H}_{x^{(1)}, x^{(2)}}^{\alpha\gamma, \beta\gamma}(R^N)$ and

$$\|Mu\|_{H_{x^{(1)}, x^{(2)}}^{\alpha\gamma, \beta\gamma}(R^N)} \leq C\mu \|u\|_{H_{x^{(1)}}^{\alpha\gamma}(R^N)}. \quad (2.51)$$

In the following section, we will demonstrate the using of the proven statements about multipliers to an initial-boundary value problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. Here we give simple examples of applications of Theorems 2, 8.

Example 1.

Let a function $u(x)$ has compact support in R^N and satisfies the Poisson equation

$$\Delta u(x) = f(x), \quad (2.52)$$

where a function $f(x)$ has compact support in R^N and satisfies Hölder condition for some single variable, for example, x_1 with an exponent $\gamma \in (0, 1)$

$$\langle f \rangle_{x_1}^{(\gamma)} = \sup_{h > 0} \frac{|f(x + h\vec{e}_1) - f(x)|}{h^\gamma} < \infty. \quad (2.53)$$

Consider all the second derivatives of $u(x)$ containing derivative with respect to x_1 . It is well known that in terms of Fourier transform we have the equality

$$\widetilde{\frac{\partial^2 u}{\partial x_k \partial x_1}}(\xi) = C \frac{\xi_k \xi_1}{\xi^2} \tilde{f}(\xi), \quad k = \overline{1, N}.$$

Since the function $\tilde{m}(\xi) = \frac{\xi_k \xi_1}{\xi^2}$ has the property $\tilde{m}(\lambda\xi) = \tilde{m}(\xi)$ for any positive λ and is smooth away from the origin, then it is easy to verify the conditions of Theorem 8. Consequently,

$$\left\langle \frac{\partial^2 u}{\partial x_k \partial x_1} \right\rangle_x^{(\gamma)} \leq C \langle f \rangle_{x_1}^{(\gamma)}, \quad k = \overline{1, N}, \quad (2.54)$$

where the Hölder constant in the left hand side of this inequality is taken with respect to all variables, not only with respect to x_1 .

Note that in estimate (2.54) only the second derivatives containing the derivative with respect to x_1 are present. This fact is essential as the following example shows. Let $\eta(x) \in C_0^\infty(R^3)$. Consider the function

$$u_1(x) = u_1(x_1, x_2, x_3) = (x_2^2 - x_3^2 + x_2x_3) \ln(x_2^2 + x_3^2) \eta(x_1, x_2, x_3).$$

It is immediately verified that the function with compact support $u_1(x)$ satisfies equation (2.52) with right-hand side satisfying (2.53). However, its second derivatives that do not contain the derivative with respect to x_1 not only do not satisfy Hölder condition, but are just unbounded in neighborhood of zero.

This example shows that condition (2.4) on a multiplier can not be generally dropped. Although the author does not know how close it is to the sharp condition.

Example 2.

It is interesting, in our opinion, to consider the following simple example for a parabolic equation. Let a function $u(x, t)$ with compact support in $R^N \times R^1$ satisfies the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f(x, t), \quad (2.55)$$

where the right hand side $f(x, t)$ with compact support satisfies the Hölder condition with respect to the variable t only with the exponent greater than $1/2$, that is

$$\langle f(x, t) \rangle_{t, R^N \times R^1}^{(\gamma)} < \infty, \quad \gamma \in \left(\frac{1}{2}, 1\right).$$

Making in (2.55) Fourier transform and denoting the variable of the Fourier transform with respect to t by ξ_0 , we obtain

$$\widetilde{\frac{\partial u}{\partial t}} = C \frac{i\xi_0}{i\xi_0 + \xi^2} \widetilde{f}(\xi, \xi_0).$$

Then it follows from Theorem 8 that

$$\left\langle \frac{\partial u}{\partial t} \right\rangle_{t, R^N \times R^1}^{(\gamma)} + \left\langle \frac{\partial u}{\partial t} \right\rangle_{x, R^N \times R^1}^{(2\gamma)} \leq C \langle f(x, t) \rangle_{t, R^N \times R^1}^{(\gamma)}.$$

In particular, since $2\gamma \in (1, 2)$, the derivative $\frac{\partial u}{\partial t}$ has derivatives with respect to x_i and

$$\sum_{i=1}^N \left\langle \frac{\partial^2 u}{\partial t \partial x_i} \right\rangle_{x, R^N \times R^1}^{(2\gamma-1)} \leq C \langle f(x, t) \rangle_{t, R^N \times R^1}^{(\gamma)}.$$

Note again that, as in the previous example, the second derivatives with respect to the variables x_i can be unbounded in general, for example, $u(x_1, x_2, t) = (x_1^2 - x_2^2 + x_2 x_1) \ln(x_1^2 + x_2^2) \eta(x_1, x_2) \psi(t)$, $\eta \in C_0^\infty(R^2)$, $\psi \in C_0^\infty(R^1)$.

3 Model problems in a half-space for the linearized Cahn-Hilliard equation with dynamic boundary conditions.

In this section we consider the Schauder estimates for initial boundary value problems in a half-space for the linearized Cahn-Hilliard equation with dynamic boundary conditions. These problems are not included in the well-known general theory of parabolic initial-boundary value problems (see, eg, [8] - [10]). However, we significantly use the results of [8].

The presentation in this section is very sketchy. More detailed text will be given in a forthcoming paper.

Define the used below space of smooth functions. Let Ω be a domain in R^N , which can be unbounded. Denote by $\Omega_T = \Omega \times (0, T)$, where $T > 0$ or $T = +\infty$. We use Banach functional spaces $C^{l_1, l_2}(\overline{\Omega_T})$ with elements $u(x, t)$, $x \in \overline{\Omega}$, $t \in [0, T]$, $l_1, l_2 > 0$ are non-integer. These spaces are defined, for example, in [36] and consist of functions with smoothness with respect to the variables x up to the order l_1 and with smoothness with respect to the variable t up to the order l_2 , ie, having a finite norm

$$|u|_{\overline{\Omega_T}}^{(l_1, l_2)} \equiv \|u\|_{C^{l_1, l_2}(\overline{\Omega_T})} \equiv |u|_{\overline{\Omega_T}}^{(0)} + \sum_{|\alpha|=[l_1]} \langle D_x^\alpha u \rangle_{x, \overline{\Omega_T}}^{(l_1-[l_1])} + \left\langle D_t^{[l_2]} u \right\rangle_{t, \overline{\Omega_T}}^{(l_2-[l_2])}. \quad (3.1)$$

Here $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multiindex, $|\alpha| = \alpha_1 + \dots + \alpha_N$, $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_N}^{\alpha_N}$, $[l]$ is the integer part of a number l , $|u|_{\overline{\Omega_T}}^{(0)} = \max_{\overline{\Omega_T}} |u(x, t)|$, $\langle D_x^\alpha u \rangle_{x, \overline{\Omega_T}}^{(l_1-[l_1])}$, $\left\langle D_t^{[l_2]} u \right\rangle_{t, \overline{\Omega_T}}^{(l_2-[l_2])}$ are Hölder constants of the corresponding functions with respect to x and t correspondingly over a domain $\overline{\Omega_T}$. Besides quantities in (3.1), for functions from the space $C^{l_1, l_2}(\overline{\Omega_T})$ the Hölder constants of the derivatives $D_x^\alpha u$ with respect to t are finite with some exponents and the same is true for the Hölder constants of the derivatives $D_t^{[l_2]} u$ with respect to x and for mixed derivatives up to some order. Estimates of all these Hölder constants are obtained by interpolation with the using of (3.1) - see, for example [36]. Below we will use the space $C^{l, l/4}(\overline{\Omega_T})$,

where l is a non-integer positive number and the norm in this space we will denote for simplicity by $|u|_{\overline{\Omega_T}}^{(l)}$.

We will use also the spaces $C_0^{l_1, l_2}(\overline{\Omega_T})$, where zero at the bottom of the notation denotes a closed subspace of $C^{l_1, l_2}(\overline{\Omega_T})$, consisting of functions whose derivatives with respect to t up to the order $[l_2]$ vanish identically at $t = 0$. The functions of these spaces can be considered to be extended identically zero to $t \leq 0$ with the preservation of the class.

We proceed to the formulation of the problem. Denote $Q_+^{N+1} = \{(x, t) \in R^N \times R^1 : x_N \geq 0, t \geq 0\}$, $Q_{+,T}^{N+1} = \{(x, t) \in R^N \times R^1 : x_N \geq 0, t \in [0, T]\}$, $Q_+^N = \{(x, t) \in R^N \times R^1 : x_N = 0, t \geq 0\}$, $Q_{+,T}^N = \{(x, t) \in R^N \times R^1 : x_N = 0, t \in [0, T]\}$, $Q^N = \{(x, t) \in R^N \times R^1 : x_N = 0\} = R^{N-1} \times R^1$, $x = (x', x_N)$. Consider in Q_+^{N+1} the following initial boundary value problem for the unknown function $u(x, t)$:

$$\frac{\partial u}{\partial t} + \Delta^2 u = f(x, t), \quad (x, t) \in Q_+^{N+1}, \quad (3.2)$$

$$\frac{\partial \Delta u}{\partial x_N} = g(x', t), \quad (x', t) \in Q_+^N, \quad (3.3)$$

$$u(x, 0) = 0, \quad x_N \geq 0, \quad (3.4)$$

$$\frac{\partial u}{\partial t} - a \Delta_{x'} u = h_1(x', t), \quad (x', t) \in Q_+^N, \quad (3.5)$$

where Δ is the Laplace operator, $\Delta_{x'}$ is the Laplace operator with respect to the variables x' , a is a positive constant, and we assume that the function $u(x, t)$ is bounded at $|x| \rightarrow \infty$. Together with boundary dynamic condition (3.5) (instead of this condition conditions) we also consider other boundary condition

$$\frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x_N} = h_2(x', t), \quad (x', t) \in Q_+^N. \quad (3.6)$$

The physical meaning of the condition of the form (3.5) is explained, for example, in [16], and the condition (3.6) is explained, for example, in [13]. In this case, in [13] was considered a more general boundary condition

$$\frac{\partial u}{\partial t} - a \Delta_{x'} u - b \frac{\partial u}{\partial x_N} = h(x', t), \quad (x', t) \in Q_+^N.$$

But (at least when considering the classes of smooth functions) the term $b \frac{\partial u}{\partial x_N}$ is in this condition a junior (in order) term and can be omitted when considering the model problem.

We assume that the given functions f, g, h_1, h_2 have compact supports and belong to the following spaces with zero at the bottom with some $\gamma \in (0, 1)$

$$f \in C_0^{\gamma, \frac{\gamma}{4}}(Q_+^{N+1}), \quad g \in C_0^{1+\gamma, \frac{1+\gamma}{4}}(Q_+^N), \quad h_1 \in C_0^{2+\gamma, \frac{2+\gamma}{4}}(Q_+^N), \quad h_2 \in C_0^{3+\gamma, \frac{3+\gamma}{4}}(Q_+^N). \quad (3.7)$$

The solution $u(x, t)$ we will suppose in the class $C_0^{4+\gamma, \frac{4+\gamma}{4}}(Q_+^{N+1})$, that is dictated by the anisotropy of equation (3.2). But besides we require that $u_t(x', 0, t) \in C_0^{2+\gamma, \frac{2+\gamma}{4}}(Q_+^N)$ or $u_t(x', 0, t) \in C_0^{3+\gamma, \frac{3+\gamma}{4}}(Q_+^N)$ depending on the type of dynamic boundary conditions.

Note that in view of (3.4), (3.2) and the fact that the given functions f, g, h_1, h_2 belong to the spaces with zero at the bottom the function $u(x, t)$ must satisfy the condition $\partial u / \partial t(x, 0) \equiv 0$. Together with (3.4) this allows to consider the function $u(x, t)$ and all the given functions f, g, h_1, h_2 to be extended by zero to $t < 0$ and consider relations (3.2) - (3.6) for all values of the time variable $t \in R^1$.

3.1 Problem (3.2) - (3.5).

Consider problem (3.2) - (3.5). Denote

$$\rho(x', t) \equiv u(x', 0, t) = u(x, t)|_{x_N=0}. \quad (3.8)$$

Condition (3.5) allows to find the value of the unknown function $u(x, t)$ at $x_N = 0$, that is the function $\rho(x', t)$, namely,

$$\rho(x', t) = \Gamma_a(x', t) * h_1(x', t), \quad (3.9)$$

where $\Gamma_a(x', t)$ is the fundamental solution of the heat operator $L_a \equiv \partial / \partial t - a \Delta_{x'}$. It is well known that expression (3.9) can be obtained from (3.5) by applying the Fourier transform with respect to the variables x' and t . In other words, denoting for a function $v(x', t)$

$$\tilde{v}(\xi_0, \xi) = \int_{-\infty}^{+\infty} dt \int_{R^{N-1}} e^{-i\xi_0 t - i\xi x'} v(x', t) dx'$$

and applying this transform to relation (3.5) (recall that all the functions are assumed to be extended by zero to $t < 0$), in view of the known properties of the Fourier transform of derivatives, we find that

$$\tilde{\rho}(\xi_0, \xi) = \frac{\tilde{h}_1(\xi_0, \xi)}{i\xi_0 + a\xi^2}. \quad (3.10)$$

Estimates for the potential in (3.9) are well known in the case when its density $h_1 \in C^{k+\gamma, \frac{k+\gamma}{2}}(Q_+^N)$. However, in our case we are dealing with another anisotropy of smoothness of the space for the density, namely $h_1 \in C_0^{2+\gamma, \frac{2+\gamma}{4}}(Q_+^N)$. Therefore

known properties of the potential for the heat operator are inapplicable in our case. Furthermore, the results of [1] and Theorem 1 also are inapplicable, since the anisotropy of homogeneity of the kernel does not coincide with the anisotropy of the smoothness of the density h_1 . Therefore, we obtain estimates of the Hölder constants for highest derivatives of the function $\rho(x', t)$ in the space $C^{4+\gamma, \frac{4+\gamma}{4}}(Q_+^N)$ using Theorem 8.

Consider first the Hölder constant in the variable t of the derivative $\rho_t(x', t)$. In view of relation (3.10) and known properties of the Fourier transform of derivatives

$$\tilde{\rho}_t = \frac{i\xi_0}{i\xi_0 + a\xi^2} \tilde{h}_1(\xi_0, \xi). \quad (3.11)$$

Consider the function

$$\tilde{m}_1(\xi_0, \xi) = \frac{i\xi_0}{i\xi_0 + a\xi^2}. \quad (3.12)$$

Evidently this function is homogeneous of degree zero

$$\tilde{m}_1(\lambda^2\xi_0, \lambda\xi) = \tilde{m}_1(\xi_0, \xi), \quad \lambda > 0. \quad (3.13)$$

Besides this function is smooth on the set $B_1 = \{(\xi_0, \xi) : 1/8 < |\xi_0| + \xi^2 < 8\}$. Therefore it is trivial to verify that $\tilde{m}_1(\xi_0, \xi)$ satisfies the condition of theorem 8. Consequently

$$\langle \rho_t \rangle_{t, Q_+^N}^{(\frac{2+\gamma}{4})} \leq C \langle h_1 \rangle_{t, Q_+^N}^{(\frac{2+\gamma}{4})}. \quad (3.14)$$

Similar estimates of other derivatives of ρ result in the estimate

$$|\rho|_{C^{4+\gamma, \frac{4+\gamma}{4}}(Q_{+,T}^N)} + |\rho_t|_{C^{2+\gamma, \frac{2+\gamma}{4}}(Q_{+,T}^N)} \leq C_T |h_1|_{C^{2+\gamma, \frac{2+\gamma}{4}}(Q_{+,T}^N)}. \quad (3.15)$$

Thus in problem (3.4)-(3.5) condition (3.5) can be replaced by the condition

$$u(x', 0, t) = \rho(x', t), \quad (3.16)$$

where for the function $\rho(x', t)$ estimate (3.15) is valid. Then from the results of [8] it follows that this problem has a unique solution $u(x, t)$ and

$$|u|_{C^{4+\gamma, \frac{4+\gamma}{4}}(Q_{+,T}^{N+1})} \leq C_T \left(|f|_{C^{\gamma, \frac{\gamma}{4}}(Q_{+,T}^{N+1})} + |g|_{C^{1+\gamma, \frac{1+\gamma}{4}}(Q_{+,T}^N)} + |h_1|_{C^{2+\gamma, \frac{2+\gamma}{4}}(Q_{+,T}^N)} \right), \quad (3.17)$$

and besides in view of (3.15)

$$|u_t(x', 0, t)|_{C^{2+\gamma, \frac{2+\gamma}{4}}(Q_{+,T}^N)} \leq C_T |h_1|_{C^{2+\gamma, \frac{2+\gamma}{4}}(Q_{+,T}^N)}. \quad (3.18)$$

Thus we have proved the following assertion.

Theorem 9 *Under conditions (3.7) and for any $T > 0$ problem (3.4)-(3.5) has the unique solution $u(x, t)$ from the space $C^{4+\gamma, \frac{4+\gamma}{4}}(Q_{+, T}^{N+1})$ and estimates (3.17), (3.18) are valid.*

3.2 Problem (3.2) - (3.4), (3.6).

As in the previous section, we reduce the problem to a problem with condition (3.16) instead of condition (3.6) after determining the function $\rho(x', t) \equiv u(x', 0, t)$ from the conditions of the problem. However, in this case the boundary operator in the left side of (3.6) is not a local operator (as opposed to (3.5)), so its consideration requires somewhat more complex reasoning. This is due to the fact that in this case a more complex multiplier arises, which is not a homogeneous function. To study this multiplier, we extract it's "the main" homogeneous part.

Using well-known results on the solvability of parabolic boundary value problems, we can reduce problem (3.2) - (3.4), (3.6) to the case when $f \equiv 0$ and $g \equiv 0$. Besides, we will denote for simplicity the function h_2 as just h .

Make in problem (3.2) - (3.4), (3.6) the Fourier transform with respect to the variables x' and t . As a result these relations take the form

$$i\xi_0 \tilde{u} + \left(-\xi^2 + \frac{d^2}{dx_N^2} \right)^2 \tilde{u} = 0, \quad x_N > 0, \quad (3.19)$$

$$\frac{d}{dx_N} \left(-\xi^2 + \frac{d^2}{dx_N^2} \right) \tilde{u} \Big|_{x_N=0} = 0, \quad (3.20)$$

$$i\xi_0 \tilde{\rho} - a \frac{d\tilde{u}}{dx_N} \Big|_{x_N=0} = \tilde{h}, \quad (3.21)$$

$$|\tilde{u}| \leq C, \quad x_N \rightarrow \infty, \quad (3.22)$$

where $\rho(x', t) \equiv u(x', 0, t)$.

From these relations we find

$$\tilde{\rho} = \frac{\tilde{h}(\xi, \xi_0)}{i\xi_0 + \frac{2\sqrt{\xi^2 + \sqrt[2]{-i\xi_0}}\sqrt{\xi^2 - \sqrt[2]{-i\xi_0}}}{\sqrt{\xi^2 + \sqrt[2]{-i\xi_0}} + \sqrt{\xi^2 - \sqrt[2]{-i\xi_0}}}} = \frac{\widetilde{h^{(k)}}(\xi, \xi_0)}{\widetilde{M}(\xi, \xi_0)}. \quad (3.23)$$

Let $\alpha = (\alpha_1, \dots, \alpha_{N-1})$, $\beta = (\beta_1, \dots, \beta_{N-1})$ be multi-indexes, $|\alpha| = 4$, $|\beta| = 3$. By simple algebraic manipulations we can get from the last equality

$$\widetilde{D_{x'}^\alpha \rho} = \frac{i\xi_l}{i\xi_0 + |\xi|} \widetilde{D_{x'}^\beta h} + \mathbb{A}\rho, \quad (3.24)$$

where \mathbb{A} is a smoothing operator.

Using assertion about multipliers from the previous section we can eventually get the estimate

$$|u|_{Q_{+,T}^{N+1}}^{(4+\gamma, \frac{4+\gamma}{4})} + |D_t u(x', 0, t)|_{Q_{+,T}^N}^{(3+\gamma, \frac{3+\gamma}{4})} \leq C_T \left(|f|_{Q_{+,T}^{N+1}}^{(\gamma, \frac{\gamma}{4})} + |g|_{Q_{+,T}^N}^{(3+\gamma, \frac{3+\gamma}{4})} + |h|_{Q_{+,T}^N}^{(3+\gamma, \frac{3+\gamma}{4})} \right). \quad (3.25)$$

Thus we have the following theorem.

Theorem 10 *Let $T > 0$ be arbitrary and let for problem (3.2) - (3.4), (3.6) conditions (3.7) are satisfied. Then this problem has the unique solution $u(x, t)$ from the space $u(x, t) \in C^{4+\gamma, \frac{4+\gamma}{4}}(Q_{+,T}^{N+1})$, $u_t(x', 0, t) \in C^{3+\gamma, \frac{3+\gamma}{4}}(Q_{+,T}^N)$ and estimate (3.25) is valid.*

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